REFERENCES

1. Oziraner, A. S. and Rumiantsev, V. V., The method of Liapunov functions in the stability problem for motion with respect to a part of the variables. PMM Vol. 36, №2, 1972.
2. Krasovskii, N. N., Stability of Motion. Stanford Univ. Press, Stanford, Cal., (Translation from Russian), 1963.
3. Rumiantsev, V. V., On asymptotic stability and instability of motion with respect to a part of the variables. PMM Vol. 35, №1, 1971.
4. Oziraner, A. S., On the stability of equilibrium positions of a solid body with a cavity containing a liquid. PMM Vol. 36, №5, 1972.
5. Fikhtengol'ts, G. M., Course on Differential and Integral Calculus, Vol. 2. Moscow, "Nauka", 1966. (See also: Fundamentals of Mathematical Analysis Vol. 2, Pergamon Press, Book №10060, 1965).
6. Oziraner, A. S., On asymptotic stability relative to a part of the variables. Vestn. Mosk. Gos. Univ., Ser. Matem. Mekhan., №1, 1972.
7. Oziraner, A. S., On certain theorems of Liapunov's second method. PMM Vol. 36, №3, 1972.
8. Risito, C., Sulla stabilità asintotica parziale. Ann. di Math. purra ed appl., Ser. 4, Vol. 84, 1970.
9. Nemytskii, V. V. and Stepanov, V. V., Qualitative Theory of Differential Equations. Princeton, N. J., Princeton Univ. Press, 1960.
10. La Salle, J. and Lefschetz, S., Stability by Liapunov's Direct Method. New York, Academic Press, 1961.
11. Barbashin, E. A., Introduction to Stability Theory. Moscow, "Nauka", 1967.
12. Malkin, I. G., Theory of Stability of Motion. Moscow, "Nauka", 1966.
13. Oziraner, A. S., On the question of motion stability relative to a part of the variables. Vestn. Mosk. Gos. Univ., Ser. Matem. Mekhan., №1, 1971.
14. Salvadori, L., Sul problema della stabilità asintotica. Meccanica. J. of Italian Association of Theor. and Appl. Mec., Vol. 7, №4, 1972.

Translated by N. H. C.

UDC 62-50

ON THE OPTIMAL SPACING OF MEASUREMENTS IN THE METHOD OF LEAST SQUARES

PMM Vol. 37, №4, 1973, №4, 666-673

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(Received December 14, 1971)

We consider the problem of the distribution of a specified number of measurments on a given interval, ensuring the least variance of the estimate of one of the parameters linearly related with the function being measured. Assuming a normal distribution law for the measurement errors, we derive equations describing necessary extremum conditions for the corresponding variance. Using

these equations we analyze the case of optimal disposition of the measurements at the ends of an interval. We investigate in more detail the case of parabolic regression, for which we establish the nature and the number of optimal points.

One of the peculiarities of the problem of optimal spacing of observations is connected with the fact that it arose as a result of applying the methods of mathematical statistics, but its resolution requires us to bring in other branches of mathematics. Thus, for examole, in [1] a bound on the maximum numer of optimal (with respect to the variance of the estimate of some parameter) observation instants was established by methods of mathematical analysis, in [2] it was shown with the aid of linear programing theory that the number of optimal points with different vector-gradients of the function being measured with respect to the parameters being estimated does not exceed the total number of parameters being determined. This proposition is proved under an essential condition on the admissibility of any noninteger values of the weighting coefficients (i.e. on the continuality of their values) with the single requirement that the sum of these coefficients equal the total number of measurements. The first investigation of this question was apparently carried out in [3] wherein the stated problem was considered for two parameters under the assumption that an arbitrarily large number of measurements can be made at a point. Papers [4, 5], using the results of [2], examined the problem of the simultaneous choice of an optimal strategy and of the composition of the measurements under certain assumptions on the nature of the correlation between them. The assertions and algorithms presented in these papers are valid, strictly speaking, under the condition that the weighting coefficient values are continual, i.e. at a sufficiently large composition of measurements. The general problem of observation was investigated in [6] from the viewpoint of Pontriagin's maximum principle as it applies to systems of linear differential equations; the measurement process is interpreted as a control process with specified constraints. Below, under the assumptions in [1], we look at certain questions connected with the problem of an optimal disposition of the measurements without the condition on the continuality of the values of the weighting coefficients. Here, however, we do not pose the problem of constructing a computation algorithm, therefore, the derived equations for the necessary conditions are not supplemented by conditions of sufficiency. These equations are used for obtaining certain results analytically; in particular, they allow a sufficiently detailed investigation of the parabolic regression case. The existence of the so-called ballast instants of measurements, i.e. those not leading to a lessening of the *a priori* variance of some parameter, is proved for this case. We show that in the given case these instants alternate with the optimal instants of measurements.

1. Equations for the optimal and the ballast instants of measurements. We assume that the linear dependency

$$y(t) = \sum_{i=1}^{m} x_i f_i(t)$$

exists between the function $y(t)$ being measured and the parameters $x_1, \ldots, x_m$ being determined. Here the $f_i(t)$ are known time functions which are assumed to be continuously differentiable on a specified interval $[t_0, T]$ wherein we seek the optimal disposition of a specified number N of measurements. If we assume that independent measurements of function $y(t)$ are made at the instants $t_1, \ldots, t_N$ and that here the error at each measurement is distributed on a normal law, then the correlation matrix of the estimates of parameters $x_1, \ldots, x_m$ found by the method of greatest likelihood, has the form

$$K = C^{-1}, \quad C = \left\| \sum_{s=1}^{N} f_k(t_s) f_r(t_s) \right\| \quad (k, r = 1, \ldots, m) \tag{1.1}$$

In this expression we have assumed also that mean and the variance of the error equal zero and unity, respectively.

We consider the problem of minimizing the variance of the estimate of some parameter x_i, i.e. the diagonal element K_{ii} of matrix K, being a function of $t_1, \ldots, t_N$. Let us find an expression for the partial derivatives of K_{ii} with respect to each of the variables $t_1, \ldots, t_N$. From relation (1.1) we have

$$\frac{\partial K}{\partial t_j} = - K \frac{\partial C}{\partial t_j} K \qquad (j = 1, \ldots, N)$$

whence it follows that

$$\frac{\partial K_{ii}}{\partial t_j} = \left(\frac{\partial K}{\partial t_j}\right)_{ii} = - \sum_{q, r=1}^{m} \left(\frac{\partial C}{\partial t_j}\right)_{qr} K_{qi} K_{ri}$$

Further, since

$$\left(\frac{\partial C}{\partial t_j}\right)_{qr} = \frac{\partial C_{qr}}{\partial t_j} = f_q^{\cdot}(t_j) f_r(t_j) + f_q(t_j) f_r^{\cdot}(t_j)$$

we have

$$\frac{\partial K_{ii}}{\partial t_j} = -2 \sum_{q=1}^{m} f_q^{\cdot}(t_j) K_{qi} \sum_{q=1}^{m} f_q(t_j) K_{qi} \tag{1.2}$$

Here and later, as needed, we assume $\det C \neq 0$.

Thus, the necessary condition for the minimum of K_{ii} are given by the equations (at least, they are at the instants $t_j \in (t_0, T)$)

$$\sum_{q=1}^{m} f_q^{\cdot}(t_j) K_{qi} \sum_{q=1}^{m} f_q(t_j) K_{qi} = 0$$

This system of equations separates into two subsystems

$$\sum_{q=1}^{m} f_q^{\cdot}(t_j) K_{qi} = 0 \tag{1.3}$$

$$\sum_{q=1}^{m} f_q(t_j) K_{qi} = 0 \tag{1.4}$$

We note that Eqs. (1.3) and (1.4) are obtained by replacing the ith row of matrix C of (1.1) by the rows of $\{f_1^{\cdot}(t_j), \ldots, f_m^{\cdot}(t_j)\}$ and $\{f_1(t_j), \ldots, f_m(t_j)\}$, respectively, and by equating the determinants of the matrices obtained to zero.

Let us show that the solution of subsystem (1.4) does not contain optimal instants, but determines those instants (we call them ballast instants) which correspond to measurements not influencing the magnitude of the variance of the estimate of parameter x_i. In accordance with (1.1) and (1.4) we have

$$\sum_{j=1}^{m} C_{jk}^{*} K_{ji} = \begin{cases} 0, & k \neq i \\ 1, & k = i \end{cases} \tag{1.5}$$

Here C_{jk}^*, an element of a matrix C^*, is obtained by eliminating in element C_{jk} of matrix C the terms corresponding to instants which satisfy equality (1 4) Obviously, $\det C^* = \det C$. Relations (1.5) can be considered as equations in the quantities K_{ji} ($j = 1,\ldots, m$). Consequently, if matrix C^* is nonsingular, which corresponds to adopting the condition $\det C \neq 0$ (here the number of ballast points should not exceed $N - m$), then the quantity K_{ii} does not depend on the ballast instants of measurements. We have thus proved the following assertions.

Theorem 1. The measurement instants satisfying condition (1.4) do not influence the magnitude of the variance of the estimate of the corresponding parameter. Under the condition $\det C \neq 0$ the number of such instants does not exceed $N - m$.

Theorem 2. The optimal measurement instants (situated in the interval (t_0, T) and minimizing the variance of the estimate of the corresponding parameter) are determined by relations (1.3).

We note that Eqs. (1.4) for the ballast instants can be derived in another way. Suppose that before the measurement there exists the correlation martix K; after measurement the correlation matrix will be

$$K_+ = K - (1 + f_*^T K f_*)^{-1} K f_* f_*^T K, \quad f_* = \left\| \begin{matrix} f_1(t^*) \\ \vdots \\ f_m(t^*) \end{matrix} \right\|$$

where t^* is a measurement instant such that

$$(K_+)_{ii} = K_{ii} - (1 + f_*^T K f_*)^{-1} J^2, \ J = \sum_{j=1}^{m} K_{ij} f_j(t^*)$$

Hence it follows that t^* is a ballast instant under the condition $J = 0$. This condition is embraced by Eqs. (1.4).

Below we present examples of the application of the relations obtained, allowing us to determine the nature of the optimal disposition of the measurement instants in a number of cases.

2. Certain conditions for the optimal disposition of measurements at the ends of a specified interval.

Example 1. We consider at first the case $m = 2$, i.e. when the function to be measured and the parameters to be determined are connected by the relation $y(t) = x_1 f_1(t) + x_2 f_2(t)$. Here the previous assumptions relative to functions $f_1(t)$, $f_2(t)$ and to the error in the measurement of $y(t)$ are preserved. The following assertion is valid.

If

$$[f_1^2(t)]^{\cdot} [f_2^2(t)]^{\cdot} < 0 \tag{2.1}$$

on the interval $[t_0, T]$, then the only optimal points for the minimization of the variance of any of the two parameters x_1, x_2 are the ends of the interval.

The proof relies directly on the application of Eqs. (1.3) which in the given case have the form

$$f_1^{\cdot}(t_k) \sum_{i=1}^{N} f_2^2(t_i) - f_2^{\cdot}(t_k) \sum_{i=1}^{N} f_1(t_i) f_2(t_i) = 0, \quad k = 1, \ldots, N \tag{2.2}$$

Having noted that as a consequence of inequality (2.1) each of the functions $f_1^{\cdot}(t)$, $f_1(t)$, $f_2^{\cdot}(t)$, $f_2(t)$ is sign-definite, this expression can be rewritten as

$$\sum_{i=1}^{N} f_2^2(t_i) - \sum_{i=1}^{N} \left\{ \frac{f_1^{\cdot}(t_k) f_2^{\cdot}(t_k) f_1(t_i) f_2(t_i)}{[f_1^{\cdot}(t_k)]^2} \right\} = 0 \qquad (k = 1, \ldots, N)$$

According to inequality (2.1) the fraction within braces is negative for any $k = 1,\ldots,N$, consequently, (2.2) does not have a solution for any $t_i \in [t_0, T]$ whatsoever. Hence it follows that the optimal disposition of the measurements is at the ends of the interval. Analogous sufficient conditions can be formulated for an arbitrary m; however, it is very cumbersome to write them in general form.

Example 2. Let us consider these conditions further for $m = 3$ and show that the following assertion is valid. If

$$S \equiv \left[\frac{f_2^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot} \left[\frac{f_3^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot} \left[\frac{f_1(t)}{f_3(t)}\right]^{\cdot} \left[\frac{f_1(t)}{f_2(t)}\right]^{\cdot} \left[\frac{f_2(t)}{f_3(t)}\right]^{\cdot} < 0 \tag{2.3}$$

on the interval $[t_0, T]$, then there are three distinct optimal points in this interval, and two of them coincide with the ends of the interval.

Using for the proof the relation between the function to be measured and the parameters to be determined $y(t) = x_1 f_1(t) + x_2 f_2(t) + x_3 f_3(t)$, we write out relations (1.3). We obtain

$$f_1^{\cdot}(t_k) Q_{11} + f_2^{\cdot}(t_k) Q_{12} + f_3^{\cdot}(t_k) Q_{13} = 0 \qquad (k = 1, \ldots, N) \tag{2.4}$$

$$Q_{11} = \sum_{i=1}^{N} f_2^2(t_i) \sum_{i=1}^{N} f_3^2(t_i) - \left[\sum_{i=1}^{N} f_2(t_i) f_3(t_i)\right]^2$$

$$Q_{12} = -\sum_{i=1}^{N} f_1(t_i) f_2(t_i) \sum_{i=1}^{N} f_3^2(t_i) + \sum_{i=1}^{N} f_1(t_i) f_3(t_i) \sum_{i=1}^{N} f_2(t_i) f_3(t_i)$$

$$Q_{13} = \sum_{i=1}^{N} f_1(t_i) f_2(t_i) \sum_{i=1}^{N} f_2(t_i) f_3(t_i) - \sum_{i=1}^{N} f_2^2(t_i) \sum_{i=1}^{N} f_1(t_i) f_3(t_i)$$

To determine the sign of Q_{12}, with due regard to inequality (2.3) we represent Q_{12} as the double sum

$$Q_{12} = -\sum_{i,j=1}^{N} [f_1(t_i) f_2(t_i) f_3^2(t_j) - f_1(t_i) f_3(t_i) f_2(t_j) f_3(t_j)] =$$

$$-\sum_{\substack{i,j=1 \\ i>j}}^{N} f_3^2(t_j) f_3^2(t_i) \left[\frac{f_1(t)}{f_3(t)}\right]_{ij}^{\cdot} \left[\frac{f_2(t)}{f_3(t)}\right]_{ij}^{\cdot} (t_i - t_j)^2$$

Here the derivative $[f_1(t) / f_3(t)]^{\cdot}$ is taken at some point t_{ij}, $t_i \leqslant t_{ij} \leqslant t_j$. The same refers also to the derivative $[f_2(t) / f_3(t)]^{\cdot}$, but, of course, the point at which it is computed can be another one.

According to inequality (2.3) the function $[f_1(t) / f_3(t)]^{\cdot} [f_2(t) / f_3(t)]^{\cdot}$ retains its sign on the interval $[t_0, T]$, therefore,

$$\operatorname{sign} Q_{12} = -\operatorname{sign}\left\{\left[\frac{f_1(t)}{f_3(t)}\right]^{\cdot} \left[\frac{f_2(t)}{f_3(t)}\right]^{\cdot}\right\} \tag{2.5}$$

Analogously we can determine that

$$\operatorname{sign} Q_{13} = -\operatorname{sign}\left\{\left[\frac{f_1(t)}{f_2(t)}\right]^{\cdot} \left[\frac{f_3(t)}{f_2(t)}\right]^{\cdot}\right\} \tag{2.6}$$

Dividing the left-hand side of Eq. (2.4) by $f_1^{\cdot}(t_k)$ and differentiating the expression obtained with respect to t_k, we find

$$Q_{12}\left[\frac{f_2^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot}_{t_k}+Q_{13}\left[\frac{f_3^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot}_{t_k}\neq 0$$

for any $t_k \in [t_0, T]$, since according to (2.3), (2.5), (2.6),

$$\operatorname{sign}\left\{Q_{12}\left[\frac{f_2^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot} Q_{13}\left[\frac{f_3^{\cdot}(t)}{f_1^{\cdot}(t)}\right]^{\cdot}\right\}=-\operatorname{sign} S=1$$

In the derivation of this equality it was taken into account that

$$\operatorname{sign}\left\{\left[\frac{f_2(t)}{f_3(t)}\right]^{\cdot}\left[\frac{f_3(t)}{f_2(t)}\right]^{\cdot}\right\}=-1$$

We note that during the differentiation Q_{11}, Q_{12}, Q_{13} can be considered as constant coefficients. In fact, if we assume that the optimal instants $t_1,\ldots,t_N$ have been found, then at least some of them should be zeros of the function

$$F(t)=Q_{11}+Q_{12}\frac{f_2^{\cdot}(t)}{f_1^{\cdot}(t)}+Q_{13}\frac{f_3^{\cdot}(t)}{f_1^{\cdot}(t)}$$

To estimate their number we can make use of differentiation of function $F(t)$. Thus, under the assumption that an optimal disposition exists, we find that Eq. (2.4) determined only one optimal instant inside the interval $[t_0, T]$. Consequently, the other two instants prove to be t_0 and T.

3. The ends of the interval for an optimal spacing of measurements in the case of parabolic regression. Let us examine the application of the relations obtained above to the problem of optimal spacing of measurements for the case when the function $y(t)$ to be measured and the parameters to be determined are related by

$$y(t)=x_0t^{u_0}+x_1t^{u_1}+\ldots+x_mt^{u_m}$$

Here $u_0,\ldots, u_m$ are positive integers ($u_0=0$ is admissible), where $u_0<u_1<\ldots<u_m$. We assume that $t_0>0$. Suppose that the optimal measurement instants $t_1,\ldots t_N$ minimizing the variance of the estimate of parameter x_j on the interval $[t_0, T]$ have been found. Then the following assertion is valid.

Theorem 3. There are m ballast points in the interval (t_0, T).

To prove this we denote

$$L_j(t)=\sum_{k=0}^{m}Q_{kj}t^{u_k},\qquad C=\left\|\sum_{q=1}^{N}t_q^{u_i+u_j}\right\|\qquad (i, j=0,\ldots,m)\quad (3.1)$$

where Q_{kj} are the cofactors of elements C_{kj} of matrix C. We introduce a functional Φ of the set of polynomials of the form

$$R(t)=\sum_{i,\,k=0}^{m}r_{ik}t^{u_i+u_k}$$

for which

$$\Phi\{R(t)\}=\sum_{i=1}^{N}R(t_i)\qquad (3.2)$$

Here $t_1,\ldots, t_N$ are the optimal measurement instants. Note that if polynomial $R(t)\geqslant 0$ for $t\in[t_0, T]$ and has $k<N$ roots in this intervals, then $\Phi\{R(t)\}>0$.

Using appropriate determinants we can show that

$$\Phi\{L_j(t)\,t^{u_s}\} = \begin{cases} 0, & 0 \leqslant s < j,\ m \geqslant s > j \\ \det C, & s = j \end{cases} \tag{3.3}$$

Here $L_j(s)$ is determined by expression (3.1). We need the next lemma in what is to follow.

Lemma. If the polynomial

$$u(t) = a_0 t^{u_0} + \ldots + a_p t^{u_p}, \qquad t > 0$$

has $k < p$ changes of sign at the points $\tau_1, \ldots, \tau_k$ ($\tau_i > 0$), then there exists a polynomial

$$w^k(t) = b_0 (t)^{u_{q_0}} + \ldots + b_k t^{u_{q_k}}$$

having precisely k real positive roots situated at those same points $\tau_1, \ldots, \tau_k$, and among the exponents one of them may be missing, for example u_j. Here the (distinct) indices $q_0, \ldots, q_k$ are chosen from the sequence $\{0, 1, \ldots, p\}$, $q_i \neq j$.

To prove this we assume that the exponents $u_{q_0}, \ldots, u_{q_k}$, among which we do not include u_j, have been determined and it remains only to find the coefficients $b_0, \ldots, b_{j-1}$, $b_{j+1}, \ldots, b_k$ of polynomial $w^k(t)$ from the system of equations

$$\sum_{i=0}^{k} b_i \tau_s^{u_{q_i}} = 0 \qquad (s = 1, \ldots, k) \tag{3.4}$$

We set coefficient b_0 equal to unity. Then, in order that this system have a solution, it is necessary and sufficient that matrices A_1 and A_2 have equal rank, where

$$A_1 = \begin{Vmatrix} \tau_1^{u_{q_1}} & \ldots & \tau_1^{u_{q_k}} \\ \ldots & \ldots & \ldots \\ \tau_k^{u_{q_1}} & \ldots & \tau_k^{u_{q_k}} \end{Vmatrix}, \qquad A_2 = \begin{Vmatrix} \tau_1^{u_{q_0}} & \tau_1^{u_{q_1}} & \ldots & \tau_1^{u_{q_k}} \\ \ldots & \ldots & \ldots & \ldots \\ \tau_k^{u_{q_0}} & \tau_k^{u_{q_1}} & \ldots & \tau_k^{u_{q_k}} \end{Vmatrix}$$

Consequently, the indices $q_0, \ldots, q_k$ must be such that this solvability condition is fulfilled. Let us show that such a choice can always be made. In fact, the points $\tau_1, \ldots, \tau_k$ are the roots of the polynomial $u(t)$, i.e.

$$\sum_{i=0}^{p} a_i \tau_s^{u_i} = 0 \qquad (s = 1, \ldots, k)$$

We divide the system by a_j and rewrite it in the form

$$\sum_{i=0}^{p} a_i' \tau_s^{u_i} = -\tau^{u_j}, \quad a_i' = \frac{a_i}{a_j} \qquad (s = 1, \ldots, k)$$

Since $\tau_1, \ldots, \tau_k$ are distinct roots of polynomial $u(t)$, the rank of the fundamental matrix of this system, considered relative to the coefficients $a_0', \ldots, a_p'$ ($a_j' = 0$), equals k. This means that we can find k columns of the fundamental matrix, from which we can set up a matrix A_1 with rank k. Since $k \leqslant p - 1$, we can find one more column, by adding which to matrix A_1 we obtain a matrix A_2 with the same rank. Consequently, the exponents $u_{q_0}, \ldots, u_{q_k}$ can be chosen such that system (3.4) has a solution and a polynomial $w^k(t)$ is defined, which by the Descartes theorem has no more than k real roots, i.e. its roots are $\tau_1, \ldots, \tau_k$; this proves the lemma.

We now assume that polynomial $L_j(t)$ has $k < m$ changes of sign. Then, according to the lemma we can set up a polynomial

$$w_j^k(t) = t^{u_{q_0}} + \sum_{i=1}^{k} b_i t^{u_{q_i}}, \quad q_i \neq j \quad (i = 0, 1, \ldots, k)$$

such that the polynomial $\bar{L}_j(t) = \vartheta L_j(t) w_j^k(t)$ is nonnegative on the interval $[t_0, T]$ with an appropriate choice of the sign of constant ϑ. Therefore, the inequality $\Phi\{\bar{L}_j(t)\} > 0$ should be fulfilled since $k < m \leqslant N$. But it follows from (3.2) that $\Phi\{\bar{L}_j(t)\} = 0$. The contradiction obtained shows that the number of changes of sign of polynomial $L_j(t)$ equals m (for $t > 0$).

From the course of the proof it is clear that $L_j(t)$ cannot vanish at the ends of interval $[t_0, T]$. Otherwise we could exclude the interval's ends in functional Φ and prove, for example, that the polynomial $L_j(t)$ has m roots on an interval $[t_1, T]$. But then it would turn out that this polynomial has $m + 1$ roots on the whole positive semiaxis $t > 0$, which contradicts the Descartes theorem. Thus, all the ballast points are found inside the measurement interval (t_0, T).

We consider the application of Theorem 3 for: (a) $u_0 = 0$, (b) $u_0 > 0$. In case (a), a corollary follows from the theorem proved.

Corollary. When $u_0 = 0$ there exist $m + 1$ distinct optimal measurement instants on the observation interval $[t_0, T]$ and, moreover, two of them correspond to the ends of this interval. These instants alternate with the ballast instants.

In fact, since polynomial $L_j(t)$ has m simple roots in interval (t_0, T), its derivative has not less than $m - 1$ simple roots in this same interval. But when $u_0 = 0$ the polynomial $L_j^{\cdot}(t)$, by the Descartes theorem, vanishes not more than $m - 1$ times, therefore, all $m - 1$ roots are located inside interval (t_0, T). Hence it follows that only the measurement interval's ends can be the two missing optimal points. On the basis of the results obtained we investigate the dependence of the accuracy of the estimates on the length of the measurement interval, which we call the base. In many applied problems we can assume that the accuracy of the estimates increases as the base grows. In the given case this assumption has a rigorous proof.

Theorem 4. For $u_0 = 0$ and for an optimal disposition of the measurements the variance of the estimates decreases with lessening t_0 and (independently) with growth of T.

To prove this we assume that the optimal measurement instants $t_0, t_1, \ldots, t_{m-1}, T$ have already been found for fixed values of t_0, T. (According to the theorem's corollary the boundary instants t_0, T are also optimal). Assuming now the quantities t_0, T as beeing variable, while $t_1, \ldots, t_{m-1}$ are constants, we compute the differential of the variance of the estimate of some parameter x_j. We obtain

$$dK_{jj} = -\frac{2}{(\det C)^2}[L_j(t_0) L_j^{\cdot}(t_0)\, dt_0 + L_j(T) L_j^{\cdot}(t)\, dT] \quad (dt_0 < 0,\ dT > 0)$$

All roots of the polynomials $L_j(t)$ and $L_j^{\cdot}(t)$ lie to the left of T, therefore, when $t = T$ these polynomials have the same sign, namely, the sign of the coefficient of the leading term. Consequently, $L_j(T) L_j^{\cdot}(T) > 0$. From the root distribution of polynomials $L_j(t)$ and $L_j^{\cdot}(t)$ it follows that

$$\operatorname{sign} L_j(t_0) = (-1)^m \operatorname{sign} L_j(T) \quad \operatorname{sign} L_j^{\cdot}(t_0) = (-1)^{m-1} \operatorname{sign} L_j^{\cdot}(T)$$

But for $t = T$, sign $L_j(t) =$ sign $L_j^{\cdot}(t)$, therefore, $L_j^{\cdot}(t_0)\ L_j(t_0) < 0$; this proves the theorem.

We now consider case (b): $u_0 > 0$.

Theorem 5. When $u_0 > 0$ the accuracy of the estimates grows monotonically with increasing T.

According to Theorem 3 the polynomial $L_j(t)$ has m simple roots in interval (t_0, T) and, consequently, the derivative $L_j^{\cdot}(t)$ has not less than $m - 1$ simple roots (lying between the roots of polynomial $L_j(t)$). It can be shown that the mth positive root of $L_j^{\cdot}(t)$, which should exist by the Descartes theorem, is smaller than the smallest positive root of polynomial $L_j(t)$. It is sufficient to consider that the function $L_j(t)$ also vanishes at point $t = 0$, therefore, between the smallest positive root of polynomial $L_j(t)$ and the point $t = 0$, a point $t = t^* > 0$, should exist where $L_j^{\cdot}(t^*) = 0$. Thus, all positive roots of polynomials $L_j(t)$ and $L_j^{\cdot}(t)$ are less than T, so that when $t = T$ both polynomials have the same sign, namely, the sign of the coefficient of the leading term. The subsequent arguments coincide with those in the proof of Theorem 4, but, in contrast to the case $u_0 = 0$, the left end of the given interval may not be an optimal instant, as a consequence of which a decrease in the value of t_0 does not always lead to a decrease in the variance of the estimates.

To illustrate this circumstance we consider an example. Let

$$y(t) = x_0 t^{u_0} + x_1 t^{u_1}, \quad u_1 > u_0 > 0, \quad t \in [t_0, T]$$

After two measurements the variance of the estimate of parameter x_0 in the optimal case is

$$\sigma^2 = \frac{\bar{t}^{2u_1} + T^{2u_1}}{(\bar{t}^{u_0}T^{u_1} - \bar{t}^{u_1}T^{u_0})^2} = \frac{1}{T^{2u_0}}\frac{1+\tau^{2u_1}}{(\tau^{u_1} - \tau^{u_0})^2}, \qquad \tau = \frac{\bar{t}}{T} \tag{3.5}$$

where $\bar{t}$ has been chosen such that for a specified T, expression (3.5) has a minimum. The equation for the extremal points of the function

$$\varphi(\tau) = \frac{1+\tau^{2u_1}}{(\tau^{u_1} - \tau^{u_0})^2}, \qquad 0 < \tau < 1$$

has the form

$$(u_1 - u_0)\tau^{2u_1} + u_1\tau^{u_1 - u_0} - u_0 = 0 \tag{3.6}$$

According to the Descartes theorem this equation has only one positive root τ^*. Since the polynomial on the left-hand side of (3.6) is less than zero for $\tau = 0$ and greater than zero for $\tau = 1$, then $0 < \tau^* < 1$. Namely, this value τ^* ensures the absolute minimum of function $\varphi(\tau)$ and, consequently, of σ^2, so that the optimal value of $\bar{t}$ is $\bar{t} = \tau^* T$.

Thus, if $\bar{t} > t_0$, then one of the optimal measurement instants is located inside the interval $[t_0, T]$; however, if $t_0 \geqslant \bar{t}$, this instant coincides with the left end of this interval. Analogous conclusions hold for the case of any m, and the quantity $\bar{t}$ can depend also on the number of measurements.

The author thanks T. M. Eneev, F. L. Chernous'ko and Iu. S. Tumashev for attention to the work.

REFERENCES

1. Rudakov, V. M., On the optimal choice of navigational observation instants. Kosmich. Issledovaniia, №3, 1969.

2. Ershov, V. G., On the optimization of a trajectory measurements program.

Kosmich. Issledovaniia, №1, 1970.
3. Elfving, G., Optimum allocation in linear regression theory. Ann. Math. Statisctics, Vol. 23, №2, 1952.
4. Bakhshiian, B. Ts. and El'iasberg, P. E., Choice of the optimal strategy for orbit determination. Avtomatika i Telemekhanika, №3, 1970.
5. Bakhshiian, B. Ts., Choice of optimal instants of independent trajectory measurements. Kosmich. Issledovaniia, №1, 1970.
6. Chernous'ko, F. L., On optimization of the tracking process. PMM Vol. 33, №1, 1969.

Translated by N. H. C.

UDC 62-50

THEOREM ON $2n$ INTERVALS IN A TIME-OPTIMAL PROBLEM WITH MAGNITUDE- AND IMPULSE-CONSTRAINED CONTROL

PMM Vol. 37, №4, 1973, pp. 675-679

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(Received June 28, 1972)

We examine controlled systems which are described by linear differential equations with constant coefficients. We assume that the controlling forces are constrained simultaneously in magnitude and in impulse. The time-optimal problem for this case was investigated, for example, in [1 – 3].

Below we prove a theorem on $2n$ intervals of constancy of the optimal control. This theorem is analogous to the theorem on n intervals given in [4, 5], which holds when the control is bounded only in magnitude.

1. Statement of the problem. We consider a controlled system described by a linear matrix differential equation with real constant coefficients

$$dx/dt = Ax + bu \tag{1.1}$$

Here $x = \| x_i \|$, $A = \| a_{ij} \|$, $b = \| b_i \|$ are matrices of order $(n \times 1)$, $(n \times n)$, $(n \times 1)$, respectively, $u = u(t)$ is a scalar piecewise-continuous time function satisfying simultaneously the two constraints

$$|u(t)| \leqslant M \qquad (M = \text{const} > 0) \tag{1.2}$$

$$\int_0^\infty |u(\tau)|\, d\tau \leqslant N \qquad (N = \text{const} > 0) \tag{1.3}$$

Constraints (1.2) and (1.3) are simultaneously present, for example, when control is effected by a jet thruster. Here inequality (1.2) corresponds to the boundedness of the fuel flow rate, while inequality (1.3) corresponds to the boundedness of the thruster's propellant capacity. We denote by Ω the set of piecewise-continuous functions $u(t)$ satisfying simultaneously inequalities (1.2) and (1.3). We examine the time-optimal problem of taking system (1.1) to the origin by means of a control $u(t) \in \Omega$.